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SHANNON'S THEOREM

Claude Elwood Shannon (1916-2001)

by Luiz Renato Gomes

APPLIED FOR DISCRETE-TIME CONTROL THEORY

Dedicated to one angel that Sir Robbie Williams has been taken…

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SHANNONS'S THEOREM

By Luiz Renato Gomes

1. INTRODUCTION

Really control system issues endeavor [make effort] to explain and get information focusing on the discrete-time control theory.
This concernment has as objective to take some remarks about the ideal magnitude of integrative piecewise element used in simulating programmes.
There are lots of ways and procedures that have been developed to make simulating arrangements more representatives in comparison with actual system.
From now, we can make a question: How to calculate an ideal integrative period of time for a great power system as a 60-cycle electrical system, for instance?
Here we are going to do some observations and after that we are going to compute and proof specific theorems that are applied to this approach.

2. ELEMENTARY SIMULATION PROBLEM

We are going to suppose that we want to simulate an elementary system as follows in figure 1. We are going to considerate the first order system.

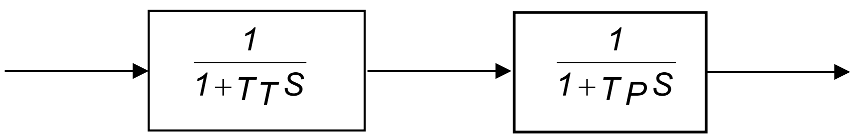


Figure 1 – An Elementary System

Considering in figure 1 T_T as the transducer time constant and T_P as the process time constant we can notice that the transducer block has the characteristic of a low-pass filter. If the value of the transducer time constant was choose without taking into account the value of the process time constant, the output signal reconstruction will not be completely performed.
This approach has been considered by lots of authors in the technical literature of automatic control and may be taken as an important detach applied for automatic control design.
Normally, the Shannon's theorem is the one that has been given the theoretical support in this way.
We confirm that the conclusion of the Shannon's theorem is very simple but previous calculus is not very easy. There are others theorems that are used to explain the Shannon's theorem and we are going to rewrite them here.

3. CAUCHY INTEGRALTHEOREM

We are going to present the Cauchy integral theorem that is based on Green's theorem that is one of the most famous theorems of vectorial calculus.
Considering an specific region ***R*** in space, a function *f(z)* is an analytic function if and only if this function is differentiable at all points of that region.
After this remark, the Cauchy theorem confirms that, for any closed contour γ completely contained in region ***R***, the following equation can be considered:

$$\oint_{\gamma} f(z)\, dz = 0 \qquad (001)$$

Taking *z* as a complex number, we have:

$$z = x + j\,y \qquad (002)$$

Rewriting function *f(z)* in a complex form too, we have:

$$f(z) = u + j\,v \qquad (003)$$

By substituting (002) and (003) into (001), gives:

$$\oint_{\gamma} f(z)\, dz = \int_{\gamma} (u + j\,v)(dx + j\,dy)$$

and then:

$$\oint_{\gamma} f(z)\, dz = \int_{\gamma} (u\,dx - v\,dy) + j\int_{\gamma} (v\,dx + u\,dy) \qquad (004)$$

We can notice that the integral symbol utilizes forms $\oint_{\gamma}$ and $\int_{\gamma}$ because the first symbol of integral is usually operating on a closed contour, considering a *simple-contour form* without double points, and the second integral symbol does not necessarily obey this rule.
From Green's theorem we write the following equation:

$$\int_{\gamma} [f(x,y)dx + g(x,y)dy] = \iint \left(\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y} \right) dx\,dy \qquad (005)$$

and

$$\int_{\gamma} [f(x,y)\,dx - g(x,y)dy] = -\iint \left(\frac{\partial g}{\partial x} + \frac{\partial f}{\partial y} \right) dx\,dy \qquad (006)$$

By comparing (004) to (006) we can write following equations considering the real part of (004):

$$f(x,y) = u \tag{007}$$

$$g(x,y) = v \tag{008}$$

and then from (007) and (008):

$$\int_{\gamma} [u\ dx - v\,dy] = -\iint \left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} \right) dx\,dy \tag{009}$$

By comparing (004) to (005) we can write following equations considering the complex part of (004):

$$f(x,y) = v \tag{010}$$

$$g(x,y) = u \tag{011}$$

and then from (010) and (011):

$$j\int_{\gamma} [v\ dx + u\,dy] = j\iint \left(\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y} \right) dx\,dy \tag{012}$$

Considering now Cauchy-Riemann equations it is possible to prove in (009) and (012) following relationships:

$$\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} = 0 \tag{013}$$

$$\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y} = 0 \tag{014}$$

so and finally

$$\oint_{\gamma} f(z)\ dz = 0 \tag{015}$$

Then, Cauchy ‘s theorem states that the integral of *f(z)* in (015) around a closed contour *Y*, considering the complex plane, is zero if and only if the function *f(z)* is analytic within and on this contour.

4. GREEN'S THEOREM

We are going to present the Green's theorem that was used as a support during the development of Cauchy's theorem.
Considering figure 2, we can notice that there exists a closed contour *R* that is a *perfect simple contour* that has the property of never being cut in more than two points by any straight that is parallel to coordinate axes neither *Y* nor *X*.

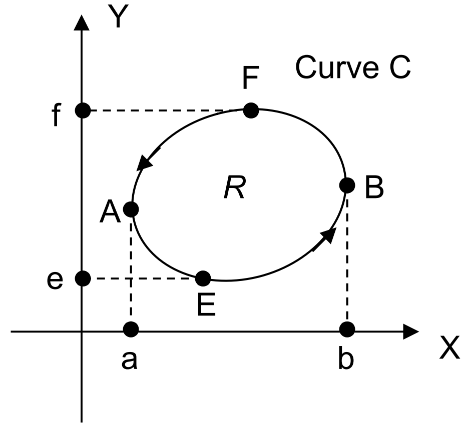


Figure 2 – A Closed Contour

Notice that curve *C* [path AFBEA] defines the region *R* that has a positive movement in a counterclockwise direction along the curve.
Be $y = Y_1(x)$ and $y = Y_2(x)$ equations of curves AEB and AFB, respectively. Considering *R* the region limited by curve *C* in figure 2 and if *M(x,y)* are analytic functions of *X* and *Y* within region *R*, we can obtain the following development.

$$\iint_R \frac{\partial M}{\partial y} dx\, dy = \int_a^b \left[\int_{Y_1(x)}^{Y_2(x)} \frac{\partial M}{\partial y} dy \right] dx = \int_a^b \left[M(x,y) \right]_{Y_1(x)}^{Y_2(x)} dx =$$

$$= \int_a^b \left[M(x,Y_2) - M(x,Y_1) \right] dx = -\int_a^b \left[M(x,Y_1) \right] dx - \int_b^a \left[M(x,Y_2) \right] dx = -\oint_C M(x,y)\, dx$$

So

$$-\oint_C M(x,y)\, dx = \iint_R \frac{\partial M(x,y)}{\partial y} dx\, dy$$

and

$$\oint_C M(x,y)\,dx = -\iint_R \frac{\partial M(x,y)}{\partial y}\,dx\,dy \qquad (016)$$

Be $x = X_1(y)$ and $x = X_2(y)$ equations of curves EAF and EBF, respectively. Considering *R* the region limited by curve *C* in figure 2 and if *N(x,y)* are analytic functions of *X* and *Y* within region *R*, we can obtain an analog development.

$$\iint_R \frac{\partial N}{\partial x}\,dx\,dy = \int_e^f \left[\int_{X_1(y)}^{X_2(y)} \frac{\partial N}{\partial x}\,dx \right] dy = \int_e^f \Big[N(x,y) \Big]_{X_1(x)}^{X_2(x)} dx =$$

$$= \int_e^f \Big[N(X_2,y) - N(X_1,y) \Big]\,dy = \int_e^f \Big[N(X_2,y) \Big]\,dy + \int_f^e \Big[N(X_1,y) \Big]\,dy = \oint_C N(x,y)\,dy$$

So

$$\oint_C N(x,y)\,dx = \iint_R \frac{\partial N(x,y)}{\partial x}\,dx\,dy \qquad (017)$$

Adding equations (016) and (017) we have:

$$\oint_C M(x,y)\,dx + \oint_C N(x,y)\,dy = -\iint_R \frac{\partial M(x,y)}{\partial y}\,dx\,dy + \iint_R \frac{\partial N(x,y)}{\partial x}\,dx\,dy$$

Joining members of previous equations we can put Green's theorem in its final form.
So

$$\oint_C \Big[M(x,y)\,dx + N(x,y)\,dy \Big] = \iint_R \left[\frac{\partial N(x,y)}{\partial x}\,dx\,dy - \frac{\partial M(x,y)}{\partial y}\,dx\,dy \right] \qquad (018)$$

We can notice that equation (018), from Green's theorem, has been used by Cauchy integral theorem development, equation (005), as the supported subside. Analog development we will obtain for equation (006).

5. CAUCHY-RIEMANN EQUATIONS

Now, we have to be able to prove other important theoretical approach based on Cauchy-Riemann equations adopted during the presentation of Cauchy integral theorem.
Let the following generic equation with the specific formulation:

$$f(x,y) = u(x,y) + j\ v(x,y) \tag{019}$$

Let be

$$z(x,y) = x + j\,y$$
$$\bar{z}(x,y) = x - j\,y \tag{020}$$

or simply

$$z = x + j\,y$$
$$\bar{z} = x - j\,y \tag{021}$$

Taking the differential elements from equations (020) we have:

$$dz = dx + j\,dy \tag{021}$$

The total derivative of function *f(x,y)* with respect to variable *z* can be computed as follows.

From equations (021) we are going to compute variables *x* and *y* as a function of variables *z* and $\bar{z}$.

By summing *z* and $\bar{z}$ we compute:

$$x = \frac{\bar{z} + z}{2} \tag{022}$$

By subtracting *z* and $\bar{z}$ we compute:

$$y = -\frac{z - \bar{z}}{2j} \tag{023}$$

The partial derivative of *x* in respect to variable *z* is:

$$\frac{\partial x}{\partial z} = \frac{1}{2} \tag{024}$$

The partial derivative of *y* in respect to variable *z* is:

$$\frac{\partial y}{\partial z} = \frac{1}{2j} \tag{025}$$

and the partial derivative of function *f* is:

$$\frac{\partial f(x,y)}{\partial z} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial z} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial z} \tag{026}$$

By substituting (024) and (025) into (026) we have:

$$\frac{\partial f}{\partial z} = \frac{\partial f}{\partial x}\left(\frac{1}{2}\right) + \frac{\partial f}{\partial y}\left(\frac{1}{2j}\right) = \frac{1}{2}\left(\frac{\partial f}{\partial x} - j\frac{\partial f}{\partial y}\right) \tag{027}$$

We have to put (027) in terms of *u* and *v*. So, we must find expressions for $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ from (019).

Taking the derivative form of (019), first with respect to variable *x* and after with respect to variable *y*, we have:

$$\frac{\partial f(x,y)}{\partial x} = \frac{\partial u}{\partial x} + j\frac{\partial v}{\partial x} \qquad 028)$$

$$\frac{\partial f(x,y)}{\partial y} = \frac{\partial u}{\partial y} + j\frac{\partial v}{\partial y} \qquad 029)$$

By substituting (028) and (029) into (027) we have:

$$\frac{\partial f}{\partial z} = \frac{1}{2}\left((\frac{\partial u}{\partial x} + j\frac{\partial v}{\partial x}) \right) - j\frac{1}{2}\left((\frac{\partial u}{\partial y} + j\frac{\partial v}{\partial y}) \right)$$

And then

$$\frac{\partial f}{\partial z} = \frac{1}{2}\left((\frac{\partial u}{\partial x} + j\frac{\partial v}{\partial x}) \right) + \frac{1}{2}\left((\frac{\partial v}{\partial y} - j\frac{\partial u}{\partial y}) \right) \tag{030}$$

Now we can make the following insight about the complex number theory. Considering the complex number, it is true that, if there is a differentiable complex function, the derivative has to there exist in all the points of this function. It is clearly that along the x-axis the operation $\frac{\partial f}{\partial y}$ is zero, in other words $\frac{\partial f}{\partial y} = 0$.

Taking the analog concerning, we can conclude that along the y-axis the operation $\frac{\partial f}{\partial x}$ is zero too, in other words $\frac{\partial f}{\partial x} = 0$.

So, we can write the following relationships for $\frac{\partial f}{\partial z}$:

in respect to x-axis

$$\frac{\partial f}{\partial z} = \frac{1}{2}\left((\frac{\partial u}{\partial x} + j\frac{\partial v}{\partial x}) \right) \tag{031}$$

in respect to y-axis

$$\frac{\partial f}{\partial z} = -j\frac{1}{2}\left((\frac{\partial u}{\partial y} + j\frac{\partial v}{\partial y}) \right) = \frac{1}{2}\left((-j\frac{\partial u}{\partial y} + \frac{\partial v}{\partial y}) \right) \tag{032}$$

It is easy to notice that equations (031) and (032) are equals to each other.
So

$$\frac{\partial f}{\partial z} = \frac{1}{2}\left((\frac{\partial u}{\partial x} + j\frac{\partial v}{\partial x}) \right) = \frac{1}{2}\left((-j\frac{\partial u}{\partial y} + \frac{\partial v}{\partial y}) \right)$$

and

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \tag{033}$$

$$\frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y} \tag{033}$$

Equations (032) and (033) are known as Cauchy-Riemann equations and they have been applied for proving Cauchy integral theorem.

6. RESIDUE THEOREM

We are going to present the residue theorem that is based on Laurent series, a particular mode of presenting an analytic function taking a sequence of numbers.
Considering a specific region ***R*** in space, a function *f(z)* is an analytic function if and only if this function is differentiable at all points of that region.
The function *f(z)* is given by Laurent series and gets the following mode:

$$f(z) = \sum_{n=-\infty}^{\infty} a_n (z - z_0)^n \tag{034}$$

From the series sequence theory, we recognize z_0 as the center of Laurent series.
Integrating term-by-term of (034) and using a closed contour *Y* encircling point z_0, we can obtain the following development:

$$\int_\gamma f(z)dz = \sum_{n=-\infty}^{\infty} a_n \int_\gamma (z - z_0)^n dz =$$

$$= \sum_{n=-\infty}^{-2} a_n \int_\gamma (z - z_0)^n dz + a_{-1} \int_\gamma (z - z_0)^{-1} dz + \sum_{n=0}^{\infty} a_n \int_\gamma (z - z_0)^n dz =$$

$$= \sum_{n=-\infty}^{-2} a_n \int_\gamma (z - z_0)^n dz + a_{-1} \int_\gamma \frac{dz}{(z - z_0)} + \sum_{n=0}^{\infty} a_n \int_\gamma (z - z_0)^n dz$$

We already know that the integral encircling a contour is always zero, but when the point of contour is a pole we have there a singularity. At this point the integral is different from zero. So, considering Cauchy integral theorem we have, for three members of previous equation that:

$$\sum_{n=-\infty}^{-2} a_n \int_\gamma (z - z_0)^n dz = 0 \tag{035}$$

$$\sum_{n=0}^{\infty} a_n \int_{\gamma} (z-z_0)^n dz = 0 \qquad (036)$$

and

$$a_{-1} \int_{\gamma} \frac{dz}{(z-z_0)} \neq 0 \qquad (037)$$

So, previous equation can be condensing in following relationship:

$$\int_{\gamma} f(z) dz = a_{-1} \int_{\gamma} \frac{dz}{(z-z_0)} \qquad (038)$$

In (038) the coefficient a_{-1} is the complex residue for pole z_0.

Taking the contour $z = z_0 + e^{jt}$ over the pole we can solve the integral (038). We can write for differential element that $dz = j e^{jt} dt$.
So

$$\int_{\gamma} \frac{dz}{(z-z_0)} = \int_0^{2\pi} \frac{j e^{jt} dt}{e^{jt}} = 2\pi j \qquad (039)$$

So we have for (038):

$$\int_{\gamma} f(z) dz = 2\pi j a_{-1} \qquad (040)$$

It is easy to notice from equation (040) that, if there are more than two poles and the contour *Y* encloses these poles of function *f(z)*, taking the encirclement in a counterclockwise direction, the integral of this function enclosed gives the summation of residues of this function *f(z)*. That is the theorem of summation of residues.
So, we have, for a generic condition, that:

$$\int_{\gamma} f(z) dz = 2\pi j \sum Residues[f(z)] \qquad (041)$$

In equation (041) $\sum Residues[f(z)]$ represents the set of residues for those poles enclosed by the contour *Y*. By the way, this theorem states that the value of a contour integral, for a

generic contour, in the complex plane, depends on the properties of a few very special points inside the contour, more properly, poles of a function in focus.
It is very important to pay attention to the fact that the positive direction in (041) is the counterclockwise direction and that the negative direction is the opposite, or the clockwise direction.

7. LAURENT SERIES

We are going to present Laurent Series, a particular form of presentation of an analytic function.
If *f(z)* is an analytic function in the annular region between and on concentric circles C_1 and C_2 , with center at point $z= z_0$ and radii r_1 and r_2 , where $r_1 > r_2$, respectively, then there exists a unique series in sequence of terms of positive and negative powers of binomial $(z- z_0)$.
Figure 3 represents two circular and concentric contours and a central point z_0.

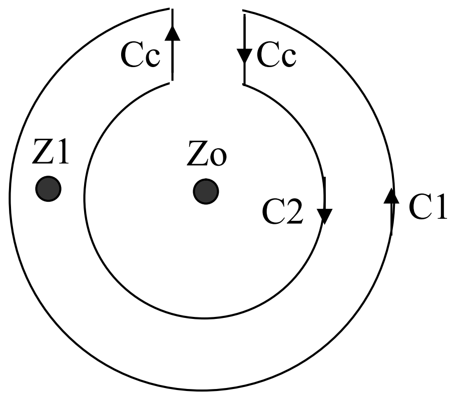


Figure 3 – Two Circular Contours

We have to notice that there have been two cut lines between circular contours C_1 and C_2 and we are going to integrate around the path $C = C_1 - C_2 + Cc - Cc$.
The contribution of *Cc* cancels one another and the final path depends upon contours C_1 and C_2. This is very interesting and important for the conclusion of this approach.
By using Cauchy integral formula *(*)* we can write the following development considering the pole Z_1:

$$f(z_1) = \frac{1}{2\pi j} \int_C \frac{f(z)}{(z - z_1)} dz \qquad (042)$$

(*) Don't confuse Cauchy integral theorem with Cauchy integral formula.

Contour C occurs around the path $C = C_1 - C_2 + Cc - C$ so that equation (042) cab be rewrite as follows:

$$f(z_1) = \frac{1}{2\pi j}\int_{C_1} \frac{f(z)}{(z-z_1)}dz + \frac{1}{2\pi j}\int_{C_C} \frac{f(z)}{(z-z_1)}dz - \qquad (043)$$

$$-\frac{1}{2\pi j}\int_{C_2} \frac{f(z)}{(z-z_1)}dz - \frac{1}{2\pi j}\int_{C_C} \frac{f(z)}{(z-z_1)}dz$$

We have to pay attention to the special condition in which the second and the fourth terms of (043) cancel one another. So, (043) can be reduced to equation (044), as follows:

$$f(z_1) = \frac{1}{2\pi j}\int_{C_1} \frac{f(z)}{(z-z_1)}dz - \frac{1}{2\pi j}\int_{C_2} \frac{f(z)}{(z-z_1)}dz \qquad (044)$$

Now, we are going to do an algebraic manipulation into (044). This changing will be done to allow facilities.

$$f(z_1) = \frac{1}{2\pi j}\int_{C_1} \frac{f(z)}{(z-z_0)-(z_1-z_0)}dz - \frac{1}{2\pi j}\int_{C_2} \frac{f(z)}{(z-z_0)-(z_1-z_0)}dz$$

We can rewrite foregoing equation as follows:

$$f(z_1) = \frac{1}{2\pi j}\int_{C_1} \frac{f(z)}{(z-z_0)(1-\frac{z_1-z_0}{z-z_0})}dz - \frac{1}{2\pi j}\int_{C_2} \frac{f(z)}{(z_1-z_0)(\frac{z-z_0}{z_1-z_0}-1)}dz$$

Foregoing equation can be rewrite as follows:

$$f(z_1) = \frac{1}{2\pi j}\int_{C_1} \frac{f(z)}{(z-z_0)(1-\frac{z_1-z_0}{z-z_0})}dz + \frac{1}{2\pi j}\int_{C_2} \frac{f(z)}{(z_1-z_0)(1-\frac{z-z_0}{z_1-z_0})}dz \qquad (045)$$

From the infinite-series theory, we can prove next relationship (046):

$$1+q+q^2+q^3+\ldots=\sum_{n=0}^{\infty} q^n=\frac{1}{1-q} \tag{046}$$

Using (046) in (045), we have for two members in right side that:

$$\frac{1}{(1-\frac{z_1-z_0}{z-z_0})}=\sum_{n=0}^{\infty}(\frac{z_1-z_0}{z-z_0})^n \tag{047}$$

$$\frac{1}{(1-\frac{z-z_0}{z_1-z_0})}=\sum_{n=0}^{\infty}(\frac{z-z_0}{z_1-z_0})^n \tag{048}$$

Substituting two foregoing equations in (045) we obtain:

$$f(z_1)=\frac{1}{2\pi j}\int_{c_1}\frac{f(z)}{(z-z_0)}\sum_{n=0}^{\infty}\left(\frac{z_1-z_0}{z-z_0}\right)^n dz+\frac{1}{2\pi j}\int_{c_2}\frac{f(z)}{(z_1-z_0)}\sum_{n=0}^{\infty}\left(\frac{z-z_0}{z_1-z_0}\right)^n dz \tag{049}$$

Considering properties from integral calculus theory, we can rewrite (049) as follows:

$$f(z_1)=\frac{1}{2\pi j}\sum_{n=0}^{\infty}\left(z_1-z_0\right)^n\int_{c_1}\frac{f(z)}{(z-z_0)^{n+1}}dz+\frac{1}{2\pi j}\sum_{n=0}^{\infty}\frac{1}{(z_1-z_0)^{n+1}}\int_{c_2}(z-z_0)^n f(z)dz$$

Rewriting previous equation, we have:

$$f(z_1)=\frac{1}{2\pi j}\sum_{n=0}^{\infty}\left(z_1-z_0\right)^n\int_{c_1}\frac{f(z)}{(z-z_0)^{n+1}}dz+\frac{1}{2\pi j}\sum_{n=0}^{\infty}(z_1-z_0)^{-n-1}\int_{c_2}(z-z_0)^n f(z)dz$$

Promoting a special changing in the variable *n*, at the second summation, in the previous equation, we have:

$$f(z_1)=\frac{1}{2\pi j}\sum_{n=0}^{\infty}\Big(z_1-z_0\Big)^{n}\int_{c_1}\frac{f(z)}{(z-z_0)^{n+1}}dz+\frac{1}{2\pi j}\sum_{n=1}^{\infty}(z_1-z_0)^{-n}\int_{c_2}(z-z_0)^{n-1}f(z)dz \tag{050}$$

Equation (050) is the generic form of Laurent series formula and can be presented in a condensed mode as follows:

$$f(z_1)=\frac{1}{2\pi j}\int_{c_1}\frac{f(z)}{(z-z_0)^{n+1}}dz\sum_{n=0}^{\infty}\Big(z_1-z_0\Big)^{n}+\frac{1}{2\pi j}\int_{c_2}(z-z_0)^{n-1}f(z)dz\sum_{n=1}^{\infty}(z_1-z_0)^{-n} \tag{051}$$

At that point we have to do a changing of variable to put equation (051) in a generic form of Laurent series formula.
So, we have:

$$f(z)=\frac{1}{2\pi j}\int_{c_1}\frac{f(v)}{(v-z_0)^{n+1}}dv\sum_{n=0}^{\infty}\Big(z-z_0\Big)^{n}+\frac{1}{2\pi j}\int_{c_2}(v-z_0)^{n-1}f(v)dv\sum_{n=1}^{\infty}(z-z_0)^{-n} \tag{052}$$

In equation (052), several authors have been used the generic formulas, as follows:

$$f(z)=\sum_{n=0}^{\infty}a_n\Big(z-z_0\Big)^{n}+\sum_{n=1}^{\infty}b_n(z-z_0)^{-n} \tag{053}$$

where:

$$a_n=\frac{1}{2\pi j}\int_{c_1}\frac{f(v)}{(v-z_0)^{n+1}}dv \tag{054}$$

$$b_n=\frac{1}{2\pi j}\int_{c_2}(v-z_0)^{n-1}f(v)dv \tag{055}$$

and inverting the signal of variable *n* and alternating the limit position at the second summation, in equation (051), we have:

$$f(z_1) = \frac{1}{2\pi j}\sum_{n=0}^{\infty}\Big(z_1 - z_0\Big)^n \int_{C_1} \frac{f(z)}{(z-z_0)^{n+1}}dz + \frac{1}{2\pi j}\sum_{n=-\infty}^{-1}(z-z_0)^n \int_{C_2}(z-z_0)^{-(n+1)}f(z)dz$$

or better

$$f(z_1) = \frac{1}{2\pi j}\sum_{n=0}^{\infty}\Big(z_1 - z_0\Big)^n \int_{C_1} \frac{f(z)}{(z-z_0)^{n+1}}dz + \frac{1}{2\pi j}\sum_{n=-\infty}^{-1}(z-z_0)^n \int_{C_2}\frac{f(z)}{(z-z_0)^{n+1}}dz$$

At that moment we have to do an important remark about the generic contours C_1 and C_2.
It is interesting for us to work with a circle C as a new path in substitution of contours C_1 and C_2 utilized in previous calculus.
And then, we have:

$$f(z_1) = \frac{1}{2\pi j}\sum_{n=-\infty}^{\infty}\Big(z_1 - z_0\Big)^n \int_{C} \frac{f(z)}{(z-z_0)^{n+1}}dz \qquad (056)$$

At that point we have to do a changing of variable to put equation (056) in a generic form of Laurent series formula.
So, we have:

$$f(z) = \frac{1}{2\pi j}\sum_{n=-\infty}^{\infty}\Big(z - z_0\Big)^n \int_{C} \frac{f(w)}{(w-z_0)^{n+1}}dw \qquad (057)$$

In equation (056), several authors have been used the generic formula, as follows:

$$f(z) = \sum_{n=-\infty}^{\infty} a_n\Big(z - z_0\Big)^n \qquad (057)$$

where:

$$a_n = \frac{1}{2\pi j}\int_{C_1} \frac{f(w)}{(w-z_0)^{n+1}}dw \qquad 058)$$

It is interesting to notice that, considering the region specified by figure 3, annular region can be expanded by increasing radius of C_1 or decreasing radius of C_2 until singularities of *f(z),* that exist outside C_1 or inside C_2, to be reached.
Paying attention to calculus, we notice too that if function *f(z)* has no singularities inside C_2 the parameter b_n is equal zero and Laurent series reduces to a Taylor's series.

8. CAUCHY INTEGRAL FORMULA

We are going to present Cauchy integral formula a different mode of calculating the value of integral along the contour *Y* enclosing a particular point *Zo* as is sketched in figure 4.

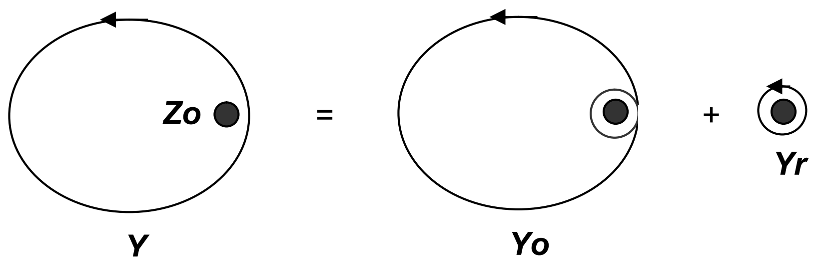


Figure 4 –Representation of Summation of Contours

In figure 4 we can notice that contour ***Y***, the original contour, is the summation of contour ***Yo*** and contour ***Yr***. Contour ***Y*** is the one that is enclosing point ***Zo***, contour ***Yo*** is the one that is not enclosing it and contour ***Yr*** is the one that is enclosing.
Contour ***Yr*** can be understood as an infinitesimal counterclockwise circle around point ***Zo*** and ***Yo*** a path that presents a cut line not enclosing this point. ***Y*** Is the total path.
We are going to take following integral to start computing the first calculus in order to prove Cauchy integral formula.

$$f(z_0) = \frac{1}{2\pi j} \oint_{\gamma} \frac{f(z)}{(z - z_0)} dz \qquad (059)$$

Taking into account that:

$$\gamma = \gamma_0 + \gamma_r \qquad (060)$$

the integral (059) gets:

$$\frac{1}{2\pi j}\oint_{\gamma}\frac{f(z)}{(z-z_0)}dz = \frac{1}{2\pi j}\oint_{\gamma_0}\frac{f(z)}{(z-z_0)}dz + \frac{1}{2\pi j}\oint_{\gamma_r}\frac{f(z)}{(z-z_0)}dz$$

Then:

$$\oint_{\gamma}\frac{f(z)}{(z-z_0)}dz = \oint_{\gamma_0}\frac{f(z)}{(z-z_0)}dz + \oint_{\gamma_r}\frac{f(z)}{(z-z_0)}dz \qquad (061)$$

From Cauchy integral theorem we know that contour integral along any path not enclosing a pole is always zero. Then, the integral over the contour ***Yo*** is zero.
So, equation (061) takes the following form:

$$\oint_{\gamma}\frac{f(z)}{(z-z_0)}dz = \oint_{\gamma_r}\frac{f(z)}{(z-z_0)}dz \qquad (062)$$

Now, we are going to represent singular point ***Zo*** in contour ***Yr*** by a complex number in polar coordinates.

Let

$$z = z_0 + r\,e^{j\theta} \qquad (063)$$

In equation (063) we have:

r : radius;
θ : angle.

and deriving (063) in respect to variable θ, gets:

$$dz = j\,r\,e^{j\theta}\,d\theta \qquad (064)$$

Substituting equations (063) and (064) into equation (062), we obtain:

$$\oint_{\gamma}\frac{f(z)}{(z-z_0)}dz = \oint_{\gamma_r}\frac{f(z_0 + r\,e^{j\theta})}{(z_0 + r\,e^{j\theta} - z_0)}\,jr\;e^{j\theta}d\theta = \oint_{\gamma_r}\frac{f(z_0 + r\,e^{j\theta})}{r\,e^{j\theta}}\,jr\;e^{j\theta}d\theta = j\oint_{\gamma_r} f(z_0 + r\,e^{j\theta})\;d\theta \qquad (065)$$

Considering the limit of radius $r \to 0$ we can rewrite (065) as follows:

$$\oint_{\gamma} \frac{f(z)}{(z-z_0)} dz = \lim_{r \to 0} j \oint_{\gamma_r} f(z_0 + r\, e^{j\theta})\, d\theta = j \oint_{\gamma_r} f(z_0)\, d\theta = jf(z_0) \oint_{\gamma_r} d\theta \qquad (066)$$

The integral from (066) goes around point ***Zo*** and, for only one encircling, allow us to write:

$$\oint_{\gamma} \frac{f(z)}{(z-z_0)} dz = jf(z_0) \oint_{\gamma_r} d\theta = 2\pi\, j\, f(z_0)$$

Basing on foregoing equation, we can finally write:

$$f(z_0) = \frac{1}{2\pi\, j} \oint_{\gamma} \frac{f(z)}{(z-z_0)} dz \qquad (067)$$

9. MAPPING THEOREM

Mapping theorem can be understood as a large application of previous theory that has been developed as a support for this real theory and others in this way.
We can start mapping theorem taking the following concernment: let function *f(s)* be a ratio of polynomials in variable *s* and consider *P* the number of poles and *Z* the number of zeros of this function. Accepting that poles and zeros of function *f(s)* lie inside a specific closed contour in the *s*-plane we have taking into account that closed contour, in the *s*-plane, maps into the *f(s)*-plane as a closed curve. The number N of clockwise encirclements of the origin of the *f(s)*-plane is equal *Z-P*, or the subtraction of the number of zeros of *f(s)* and the number of poles of it.
To prove the mapping theorem we are going to put in evidence previous concernments about Cauchy integral theorem and residue theorem. These two theorems were already proved and are the mathematical support for our conclusions.
We are going to suppose a specific function *f(s)* in the complex variable *s* of Laplace given by (068).

$$f(s) = \frac{(s+z_1)^{k1}}{(s+p_1)^{m1}} x(s) \qquad (068)$$

In equation (068) *X(s)* is the input function that is analytic in the considered closed contour and the transfer function has one zero Z_1 and one pole P_1.
Now, the derivative form of (068) is:

$$f'(s) = [\frac{(s+z_1)^{k1}}{(s+p_1)^{m1}}]'\,x(s) + \frac{(s+z_1)^{k1}}{(s+p_1)^{m1}}\,x'(s) \tag{069}$$

Dividing member to member of (069) by expression of *f(s)* from (067) we can write the following expression:

$$\frac{f'(s)}{f(s)} = \frac{[\frac{(s+z_1)^{k1}}{(s+p_1)^{m1}}]'}{[\frac{(s+z_1)^{k1}}{(s+p_1)^{m1}}]}\frac{x(s)}{x(s)} + \frac{\frac{(s+z_1)^{k1}}{(s+p_1)^{m1}}}{\frac{(s+z_1)^{k1}}{(s+p_1)^{m1}}}\frac{x'(s)}{x(s)} = \frac{[\frac{(s+z_1)^{k1}}{(s+p_1)^{m1}}]'}{[\frac{(s+z_1)^{k1}}{(s+p_1)^{m1}}]} + \frac{x'(s)}{x(s)} \tag{070}$$

The first parcel of second member of (070) can be developed as follows, taking its derivative function in respect to variable *s*:

$$\frac{[\frac{(s+z_1)^{k1}}{(s+p_1)^{m1}}]'}{[\frac{(s+z_1)^{k1}}{(s+p_1)^{m1}}]} = \frac{k_1(s+z_1)^{k1-1}(s+p_1)^{m1} - m_1(s+p_1)^{m1-1}(s+z_1)^{k1}}{\frac{(s+z_1)^{k1}}{(s+p_1)^{m1}}(s+p_1)^{2m1}} =$$

$$= \frac{k_1(s+z_1)^{k1-1}(s+p_1)^{m1}}{\frac{(s+z_1)^{k1}}{(s+p_1)^{m1}}(s+p_1)^{2m1}} - \frac{m_1(s+p_1)^{m1-1}(s+z_1)^{k1}}{\frac{(s+z_1)^{k1}}{(s+p_1)^{m1}}(s+p_1)^{2m1}} \tag{071}$$

Foregoing expression can be condensed as follows:

$$\frac{[\frac{(s+z_1)^{k1}}{(s+p_1)^{m1}}]'}{[\frac{(s+z_1)^{k1}}{(s+p_1)^{m1}}]} = \frac{k_1}{s+z_1} - \frac{m_1}{s+p_1} \tag{072}$$

By substituting (072) in (070) we can obtain:

$$\frac{f'(s)}{f(s)} = \frac{k_1}{s+z_1} - \frac{m_1}{s+p_1} + \frac{x'(s)}{x(s)} \tag{073}$$

We can notice that equation (073) is a particular case considering equation (068).
Extending the case for second order condition when there exist two zeros and two poles we have:

$$f(s) = \frac{(s+z_1)^{k1}}{(s+p_1)^{m1}} \frac{(s+z_2)^{k2}}{(s+p_2)^{m2}} x(s) \tag{074}$$

Equation (074) has to some extent a certain credit to allow an expansion to higher order in terms of presenting zeros and poles in polynomial division.
Now, the derivative form of (074) is:

$$f'(s) = [\frac{(s+z_1)^{k1}}{(s+p_1)^{m1}} \frac{(s+z_2)^{k2}}{(s+p_2)^{m2}}]' x(s) + \frac{(s+z_1)^{k1}}{(s+p_1)^{m1}} \frac{(s+z_2)^{k2}}{(s+p_2)^{m2}} x'(s) \tag{075}$$

In so doing, as in previous development, dividing member to member of (075) by expression of *f(s)* from (074) we can write the following expression:

$$\frac{f'(s)}{f(s)} = \frac{[\frac{(s+z_1)^{k1}}{(s+p_1)^{m1}} \frac{(s+z_2)^{k2}}{(s+p_2)^{m2}}]'}{[\frac{(s+z_1)^{k1}}{(s+p_1)^{m1}} \frac{(s+z_2)^{k2}}{(s+p_2)^{m2}}]} \frac{x(s)}{x(s)} + \frac{\frac{(s+z_1)^{k1}}{(s+p_1)^{m1}} \frac{(s+z_2)^{k2}}{(s+p_2)^{m2}}}{\frac{(s+z_1)^{k1}}{(s+p_1)^{m1}} \frac{(s+z_2)^{k2}}{(s+p_2)^{m2}}} \frac{x'(s)}{x(s)} \tag{076}$$

Separating each term of derivative, we can obtain:

$$\frac{f'(s)}{f(s)} = \frac{[\frac{(s+z_1)^{k1}}{(s+p_1)^{m1}}]' \frac{(s+z_2)^{k2}}{(s+p_2)^{m2}}}{[\frac{(s+z_1)^{k1}}{(s+p_1)^{m1}} \frac{(s+z_2)^{k2}}{(s+p_2)^{m2}}]} + \frac{\frac{(s+z_1)^{k1}}{(s+p_1)^{m1}} [\frac{(s+z_2)^{k2}}{(s+p_2)^{m2}}]'}{[\frac{(s+z_1)^{k1}}{(s+p_1)^{m1}} \frac{(s+z_2)^{k2}}{(s+p_2)^{m2}}]} + \frac{x'(s)}{x(s)} \tag{077}$$

Promoting the regular simplifications we have:

$$\frac{f'(s)}{f(s)} = \frac{[\frac{(s+z_1)^{k1}}{(s+p_1)^{m1}}]'}{[\frac{(s+z_1)^{k1}}{(s+p_1)^{m1}}]} + \frac{[\frac{(s+z_2)^{k2}}{(s+p_2)^{m2}}]'}{[\frac{(s+z_2)^{k2}}{(s+p_2)^{m2}}]} + \frac{x'(s)}{x(s)} \qquad (078)$$

Each parcel of second member of (077) is exactly equal to the development of (072) that was done previously. Now, we can obtain:

$$\frac{f'(s)}{f(s)} = \frac{k_1}{s+z_1} + \frac{k_2}{s+z_2} - \frac{m_1}{s+p_1} - \frac{m_2}{s+p_2} + \frac{x'(s)}{x(s)} \qquad (079)$$

It is easy to notice that a simple analog reasoning is only necessary to obtain a generic formula considering a generic function. So, we can take the following conclusion.

For a generic expression of *f(s)*, as follows:

$$f(s) = \frac{(s+z_1)^{k1}(s+z_2)^{k2}(s+z_3)^{k3}\cdots}{(s+p_1)^{k1}(s+p_2)^{k2}(s+p_3)^{k3}\cdots} x(s) \qquad (080)$$

we can obtain the following development:

$$\frac{f'(s)}{f(s)} = [\frac{k_1}{s+z_1} + \frac{k_2}{s+z_2} + \frac{k_3}{s+z_3} + \cdots] - [\frac{m_1}{s+p_1} + \frac{m_2}{s+p_2} + \frac{m_3}{s+p_3} + \cdots] + \frac{x'(s)}{x(s)} \qquad (081)$$

It is interesting to take into account that from equation (080), where the expression represents a generic condition, the main development results in a generic expression that is computed from parameters k_i and m_i with i =0,1,2,3,…

These parameters are called residues of polynomial ratio $\frac{f'(s)}{f(s)}$ and have an important concept within control theory.

We see that by taking the polynomial ratio $\frac{f'(s)}{f(s)}$, a generic zero or a generic pole of function *f(s)* become simple pole of $\frac{f'(s)}{f(s)}$.

For simplicity, we have to take, for analysis, not equations (073) or (081), but equation (082), a particular and simple case of both equations that have already been presented, with only a zero $-z_1$..

$$\frac{f'(s)}{f(s)} = \frac{k_1}{s+z_1} + \frac{x'(s)}{x(s)} \tag{082}$$

Considering, for simplicity, that the term $\frac{x'(s)}{x(s)}$ on the right-hand side of equation (082) contains no poles or zeros in the closed contour in the considered *s*-plane.
Taking the integral of both sides of (082), we have:

$$\oint \frac{f'(s)}{f(s)} ds = \oint \frac{k_1}{s+z_1} ds + \oint \frac{x'(s)}{x(s)} ds \tag{083}$$

Paying attention to equation (083) we conclude, by utilizing Cauchy theorem presented previously, see item 3, that the second parcel of second term is equal to zero. So:

$$\oint \frac{f'(s)}{f(s)} ds = \oint \frac{k_1}{s+z_1} ds \tag{083}$$

By applying residue theorem, presented previously too, see item 6, we so conclude, taking that the positive direction of angle accounting is a counterclockwise direction:

$$\oint \frac{f'(s)}{f(s)} ds = -2\pi\, j\, k_1 \tag{084}$$

Referring to generic equation (081) and taking into account that the polynomial ratio $\frac{x'(s)}{x(s)}$ is analytic in the closed contour considered here, we can notice that the factors inside the parenthesis are all simple poles located in the contour. So, taking analogous action, we have:

$$\oint \frac{f'(s)}{f(s)} ds = -2\pi\, j\, [(k_1 + k_2 + k_3 + \cdots) - (m_1 + m_2 + m_3 + \cdots)] \tag{085}$$

Observing equation (085), it is interesting to notice that, residues, here represented by $k_1, k_2, k_3, \cdots$ and $m_1, m_2, m_3, \cdots$, inside parenthesis, can taken as the exponents or grades of multiplicity of each groups of poles and zeros of the polynomial ratio $\frac{x'(s)}{x(s)}$. So $(k_1 + k_2 + k_3 + \cdots)$ represents the summation of numbers of zeros, included its multiplicities, and $(m_1 + m_2 + m_3 + \cdots)$ represents the summation of numbers of poles, included its multiplicities.

So, we can write for zeros:

$$Z = (k_1 + k_2 + k_3 + \cdots)$$

and for poles:

$$P = (m_1 + m_2 + m_3 + \cdots)$$

So, finally, taking previous equations:

$$\oint \frac{f'(s)}{f(s)} ds = -2\pi\, j\,[Z - P] \qquad (086)$$

Since *f(s)* is a complex number with magnitude $\left|f(s)\right|$ and angle θ, then:

$$f(s) = \left|f(s)\right| e^{j\theta(s)} \qquad (087)$$

The ratio $\frac{f(s)'}{f(s)}$ can be put in function of logarithmic. So then:

$$\frac{f'(s)}{f(s)} = \frac{d\ [\ln f(s)]}{ds} \qquad (088)$$

Considering, now, a specific property of logarithms, from (087), we have:

$$\ln f(s) = \ln \left|f(s)\right| + j\theta(s) \qquad (089)$$

By deriving (089) in respect to variable *s*, we have:

$$\frac{d}{ds}\ln f(s) = \frac{d}{ds}\ln\left|f(s)\right| + \frac{d}{ds}j\theta(s)$$

or better:

$$\frac{f'(s)}{f(s)} = \frac{d}{ds}\ln\left|f(s)\right| + j\frac{d\theta(s)}{ds} \tag{090}$$

Now, we are going to compute the circular integral of (090), in so doing, we can bring the whole concepts about this theory until here developing.
If the closed contour in the *s*-plane is mapped into the closed contour, considered here, we obtain the integral of (090) as follows:

$$\oint\frac{f'(s)}{f(s)}ds = \oint_{\gamma}\frac{d}{ds}\ln\left|f(s)\right|ds + j\oint_{\gamma}\frac{d\theta(s)}{ds}ds = \oint_{\gamma}d[\ln\left|f(s)\right|\,] + j\oint_{\gamma}d\theta(s) \tag{090}$$

and

$$\oint\frac{f'(s)}{f(s)}ds = \oint_{\gamma}d[\ln\left|f(s)\right|\,] + j\oint_{\gamma}d\theta(s) \tag{091}$$

In the equation (091), we can notice that term $\oint_{\gamma}d[\ln\left|f(s)\right|\,]$ is always zero, because it represents the circular integral, over a closed contour, of an analytic function, which the value is the same at the initial until the final point of the contour Y.

So we have:

$$\oint\frac{f'(s)}{f(s)}ds = j\oint_{\gamma}d\theta(s) \tag{092}$$

Now, observing the circular integral acting over the closed contour in equation (092), we can obtain the solving of previous integral as follows, taking that difference between angles $[\theta_1 - \theta_2]$ represents the difference between initial angle θ_1 and the final angle θ_2 inside the movement. So, we have:

$$\oint\frac{f'(s)}{f(s)}ds = -j[\theta_1 - \theta_2] \tag{093}$$

By comparing the result of equation (093) with the one of equation (086) we have:

$$-j[\theta_1 - \theta_2] = -2\pi\, j\,[Z - P]$$

or finally:

$$[\theta_2 - \theta_1] = 2\pi\,[P - Z]$$

or better:

$$\frac{[\theta_2 - \theta_1]}{2\pi} = [P - Z] \tag{094}$$

Paying attention to equation (094) and taking into account that the difference between values of initial angle θ_1 and final angle θ_2 ,of polynomial ratio $\frac{f'(s)}{f(s)}$, is a representative movement of it along the closed contour, considered here, it is easy to consider that the principal movement occurs around the origin, when we define the *s*-plane as a reference.
It is a fact that our choice for function *f(s)* was a complex number type $f(s) = \left|f(s)\right| e^{j\theta(s)}$ whose position depends on the *s*-plane origin.
So, we can write a new interpretation for equation (094) where it is computed the number of clockwise encirclements *N* of origin in the *s*-plane by polynomial ratio $\frac{f'(s)}{f(s)}$, as follows:

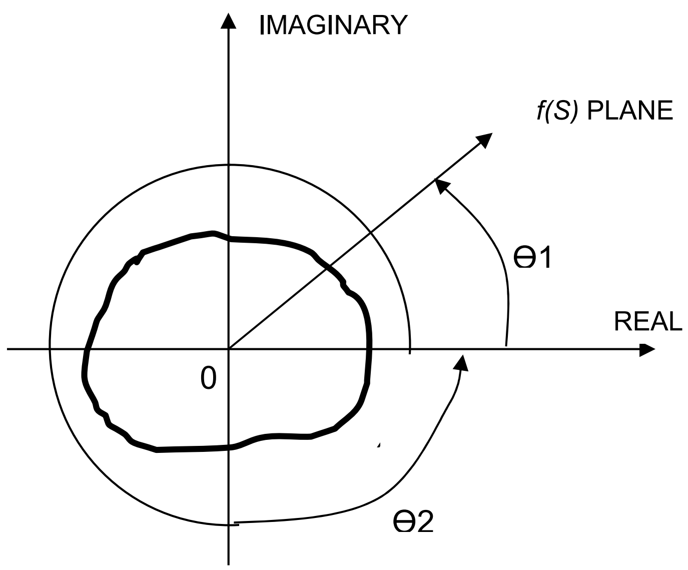


Figure 5 – Representation of Origin Encirclement

$$\frac{[\theta_2 - \theta_1]}{2\pi} = -N \qquad (095)$$

Figure 5 sketches the representative conformal movement around the origin by polynomial ratio $\frac{f'(s)}{f(s)}$.

We can notice that the mapping theorem does not need the exact number of zeros and poles, but it allow to compute only the difference between number of zeros and poles. Referring to the closed contour encircling the origin, the encirclement number will always be proportional to 2π. Now, when the origin is not encircling by the closed contour, because of being equals initial angle and the final angle, the encirclement number will always be zero.

10. SAMPLING THEOREM

Shannon's theorem or sampling theorem is very important because of the basic material for analyzing discrete-time control systems or sampled-data control systems, nowadays very large applicable systems based on computer techniques.

Sampling theorem simply defines an important condition whose idea is applicable to discrete-time systems since these systems are giving the minimum sampling frequency that is necessary to reconstruct the original signal from a specific sampled signal.

During the designing stage of any discrete-time control system, designers have to worry about filters and transducers for sampled data external signals. We can normally observe that, for the most part of designers, response time constants that have been involved with automatic control are estimated taking any known minimum time constant belongs to the process and associated with the maximum natural frequency.

This procedure, in the most part of designs, can bring some problems as excessive computation time applied for CPU [central processing unit] and others as a poor representation of system behavior during the simulation stage.

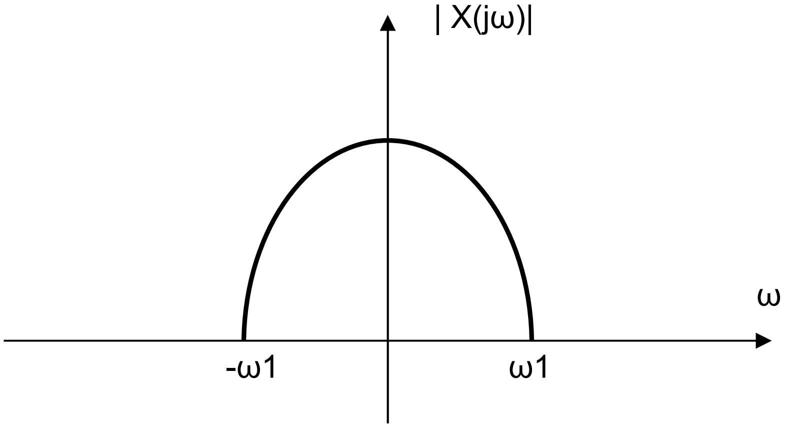


Figure 6 – Frequency Spectrum for Study

For proving sampling theorem, first, we are going to assume that a continuous signal *x(t)*, a time function, has its frequency spectrum response as shown in figure 6.
We have to consider that in this signal there are no components with frequency above or bellow frequency ω_1. So, there are no harmonic components in sampled signal here considered, only the fundamental frequency.
For continuing the development, focusing on sampling theorem, we have to do a lot of auxiliary insights based on previous calculus that have been taking as a large support.
Sampling theorem states that sampling frequency $\omega_S = \frac{2\pi}{T}$, where *T* is sampling period, has to be greater than $2\omega_1$, taking into account figure 6, or in other words:

$$\omega_S > 2\,\omega_1 \tag{096}$$

In equation (096), exactly value of $2\,\omega_1$ corresponds to the frequency spectrum of the continuous signal *x(t)*, shown in figure 6, what demonstrates that the whole signal can be reconstructed completely from the sampled signal that we are going to call *x*(t)*, only to give a different symbol or a different name, avoiding any confusion.
Now, in terms of mathematical modeling, we have to present a new concept about sampled signals and how we can work with them. A question of working with pulsing signals in *s*-plane, or complex plane, is shown by several authors and, only for us, it will need all concepts and theory that was developed before in this technical article.
Laplace transformation of any time function *x(t)* is *x(s)* and demands that function be conform with the final value theorem. This theorem states that:

$$\lim_{S \to \infty} [s\,x(s)] = 0 \tag{097}$$

In (097) *s* is the Laplace complex operator.

Initially, we are going to compute the value for Laplace transformation of a particular function that depends on the impulsive function $\delta(t - kT)$.
Equation (098) represents what mathematical observers have been used to solve a question about application of Laplace transformation techniques to sampled functions.
Normally, in ordinary continuous mathematics, we can write a Laplace transformation of a unit impulsive function as follows:

$$£[\delta(t - kT)] = e^{-kTS} \tag{098}$$

From previous equation (098) with can write:

$$£[\sum_{k=0}^{\infty} \delta(t - kT)] = 1 + e^{-TS} + e^{-2TS} + e^{-3TS} + \cdots \tag{099}$$

It is very interesting and important to know the foregoing development in sequence what admits considering a computation of each sampling of sampled signal, mathematically, expressed by a orderly summation that depends on the sampling timing pulse, here represented by a singular impulsive function.
From previous equation (046), we can rewrite (099) as follows:

$$£[\sum_{k=0}^{\infty}\delta(t-kT)] = 1+e^{-TS}+e^{-2TS}+e^{-3TS}+\dots = \frac{1}{1-e^{-TS}} \qquad (100)$$

We can notice that a series, whose sequence is defined by only a polynomial ratio, has a particular characteristic of being a finite series, in other words this series has convergence.
We are going to created a new impulsive function that will represent a modulated function that will be called *x*(t)*, so that a continuous function *x(t)* was modulated by impulsive function $[\delta(t-kT)$.
So, we can write:

$$x^*(t) = \sum_{k=0}^{\infty} X(t).\delta(t-kT)] \qquad (101)$$

Control literature authors suggest taking the following relationship to represent Laplace transformation for any sampled function *x(t)* and it will be write as follows:

$$X^*(s) = £\,[x^*(t)] = £\,[\sum_{k=0}^{\infty} x(t).\delta(t-kT)] \qquad (102)$$

In equation (102) there exists a product of two functions and for obtaining the Laplace transformation of a product of two functions, as that, it is necessary developing a new procedure considering properties of Laplace transformation theory.
Now, we can perform the Laplace transform of the product of two functions *f(t)* and *g(t)* as follows:

$$£\,[f(t)g(t)] = \int_0^{\infty}[f(t)g(t)]e^{-st}dt \qquad (103)$$

Laplace anti-transform, of only function *f(t)* , is the new function *F(s)* and given by (104) for $t>0$:

$$[f(t)] = \frac{1}{2\pi j}\int_{c-j\infty}^{c+j\infty} F(s)e^{st}\,ds \qquad (104)$$

In equation (104) c has to be understood as an abscissa convergence parameter that has been used by taking a natural convergence of functions during the Fourier transformation process. See reference 14 for more information.

Considering both functions:

$$£\,[f(t)g(t)] = \int_0^{\infty}[\frac{1}{2\pi j}\int_{c-j\infty}^{c+j\infty} F(s)e^{st}\,ds][g(t)]e^{-st}\,dt \qquad (105)$$

For maintaining the identity of each differential variable into the integration process and no making confusion, we are going to modify the complex variable *s* in both of parcels of (105). So, we have a new composition of integral as follows:

$$£\,[f(t)g(t)] = \int_0^{\infty}[\frac{1}{2\pi j}\int_{c-j\infty}^{c+j\infty} F(P)e^{Pt}\,dP\,][g(t)]e^{-st}\,dt$$

Putting each integral in adequate form, we obtain:

$$£\,[f(t)g(t)] = \frac{1}{2\pi j}[\int_{c-j\infty}^{c+j\infty} F(P)dP][\int_0^{\infty}[g(t)]e^{(P-S)t}\,dt] \qquad (106)$$

The foregoing manipulation characterized by a displacement of complex variable *p* to inside the differential in *dt* is possible since, after operations, the second integral in differential *dp* operates over it. Other consideration made by reference 1 is that because of uniform convergence of both integrals, we can invert the order of integration. Somehow, both of observations are available.

We can notice that the second integral of (106) is

$$\int_0^{\infty}[g(t)]e^{(P-S)t}\,dt = \int_0^{\infty}[g(t)]e^{-(S-P)t}\,dt = G(S-P) \qquad (107)$$

So, equation (106), from (107), gets:

$$\pounds\ [f(t)g(t)] = \frac{1}{2\pi\, j}\left[\int_{c-j\infty}^{c+j\infty} F(P)dP\right] G(S-P) \tag{108}$$

Or better:

$$\pounds\ [f(t)g(t)] = \frac{1}{2\pi\, j}\int_{c-j\infty}^{c+j\infty} F(p)G(S-P)dP \tag{109}$$

Equation (109) states that, the Laplace transform of the product of two functions *f(t)* and *g(t)* as equation (102) is given by (109), or condensing:

$$\begin{aligned} X^*(s) &= \pounds\ [x^*(t)] = \pounds\left[\sum_{k=0}^{\infty} x(t).\delta(t-kT)\right] = \\ &= \pounds\ \left[x(t)\sum_{k=0}^{\infty}.\delta(t-kT)\right] = \frac{1}{2\pi\, j}\int_{c-j\infty}^{c+j\infty}\left[X(P)\sum_{k=0}^{\infty}\delta(S-P)\right]dP \end{aligned} \tag{110}$$

Considering equation (100) we have for (110):

$$X^*(s) = \pounds\ \left[x(t)\sum_{k=0}^{\infty}\delta(t-kT)\right] = \frac{1}{2\pi\, j}\int_{c-j\infty}^{c+j\infty}\left[X(P)\,\frac{1}{1-e^{-T(S-P)}}\right]dP$$

or better:

$$X^*(s) = \frac{1}{2\pi\, j}\int_{c-j\infty}^{c+j\infty}\frac{X(P)}{1-e^{-T(S-P)}}dP \tag{111}$$

Remarks about equation (111) are important at that point of study. We always may assume or demand that poles of function *X(s)* be found lying in the left-half complex *s*-plane and that the proper function can be expressed as a ratio of two analytic polynomials whose a denominator be a polynomial of second order, at least, being true the following relationship, here copied from equation (097):

$$\lim_{P \to \infty} [P\,x(P)] = 0$$

Equation (111) is a generic expression that has been developed from specific conditions where we have been introduced an important concept of a Laplace transformation applied to a generic sampled function modulated by a generic impulsive function delta.
Now we are going to evaluate equation (111) to obtain the recursive formula that will be applied to Shannon's theorem.

Considering (111), poles of the parcel $\frac{1}{1-e^{-T(S-P)}}$ are zeros of numerator expression $1-e^{-T(S-P)}$ and solving this equation:

$$1-e^{-T(S-P)} = 0 \tag{112}$$

we have solution values of complex variable P as a function of another complex variable s as follows:

$$-T(S-P) = -j2k\pi \quad \text{Radians}$$

and

$$P = S + j\frac{2\pi}{T}k \tag{113}$$

or in generic form:

$$P = S \pm j\frac{2\pi}{T}k \quad \text{for} \quad k = 0,1,2,3,\cdots \tag{114}$$

Through equation (114) we may conclude that there exist an infinite number of poles that solve the problem.
So, in order to evaluate the integral of (111), we have to choose and define a contour taking into account previous theory applying a contour technique that was already developed on this technical article.
In figure 7, we represent the complex p-plane with set of poles of $X(s)$ and $\frac{1}{1-e^{-T(S-P)}}$.
The specified contour encloses all poles of function $\frac{1}{1-e^{-T(S-P)}}$, but it does not enclose poles of function $X(s)$.
The contour, which is sketched here, has as a particularity the line parallel to imaginary axis from inferiority level $c-j\infty$ to superiority level $c+j\infty$. They are points that have been

mapped at the infinite and, to complete the closed contour, we are going to create a semicircle Γ that has an infinite radius to meet the infinite line. So, we have figure 7

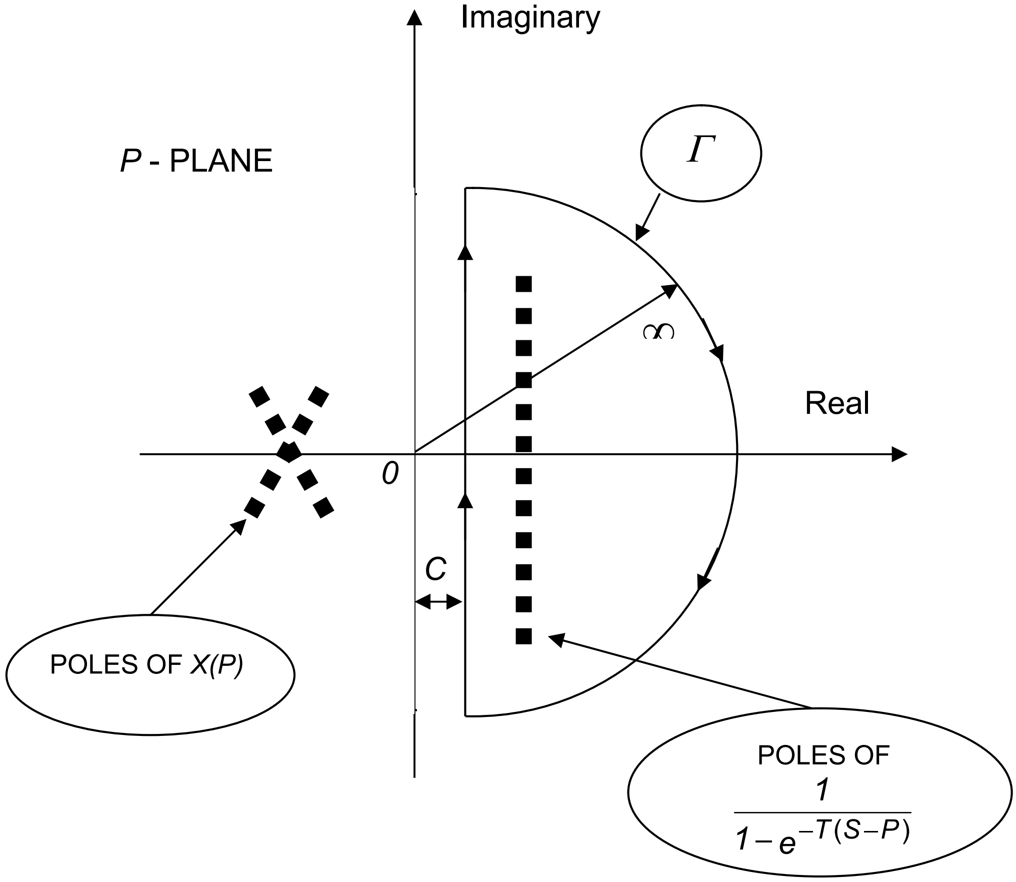


Figure 7 – Complex *P* -Plane

From figure 7, considering complex *p*-plane, we are going to promote the calculation of integral in equation 111, taking into account contours sketched:

$$X^*(s) = \frac{1}{2\pi j}\int_{c-j\infty}^{c+j\infty} \frac{X(P)}{1-e^{-T(S-P)}}dP = \\ = \frac{1}{2\pi j}\oint \frac{X(P)}{1-e^{-T(S-P)}}dP - \frac{1}{2\pi j}\int_{\Gamma} \frac{X(P)}{1-e^{-T(S-P)}}dP \quad (115)$$

Taking into account the theory, we can notice that the second integral, integral acting over semicircle Γ , does not encircle any poles and since the integrand obey the condition $\lim_{P\to\infty} [P\, x(P)] = 0$, where the degree of the denominator of *X(P)* be, at least, 2 greater than the degree of numerator, the value of this integral is zero, or:

$$\frac{1}{2\pi j}\int_{\Gamma}\frac{X(P)}{1-e^{-T(S-P)}}dP = 0 \tag{116}$$

So, we have as solution of integral:

$$X^*(s) = \frac{1}{2\pi j}\oint\frac{X(P)}{1-e^{-T(S-P)}}dp \tag{117}$$

And then, the integral along the closed contour that is encircling poles of function $\frac{1}{1-e^{-T(S-P)}}$ is different from zero and can obtained considering, now, theorem of residues. So, the solution of integral will be done by the following mathematical expression, basing on previous equation (041):

$$\oint\frac{X(P)}{1-e^{-T(S-P)}}dp = -2\pi j\sum\left[\mathrm{Re}\,sidues\ of\ \frac{X(P)}{1-e^{-T(S-P)}}\right] \tag{118}$$

From now, it is important to take care of understanding that complex variable *P* is the present variable and that the solution for the problem, initially, attained specific values computed through equation (114) when $P = S \pm j\frac{2\pi}{T}k$ for $k = 0,1,2,3,\cdots$.
Initial conditions for the problem have as the solution the consideration of poles of function $\frac{1}{1-e^{-T(S-P)}}$ lie in the left-side complex *P*-plane, or, by considering a generically case, the complex *s*-plane.
Although, we can notice, in this way, that during the process of encircling poles, by the contour line in process of integration, singular points as poles and zeros, for instance, are treated as singularities and cannot seen as analytic points for what calculus are available. That is what, during encircling process, these points are at least involved by a contour line, or better, the mapping does not pass through singularities.
And then, we can obtain, for computing residues, the following relationship:

$$\mathrm{Re}\,sidues\ of\ \frac{X(P)}{1-e^{-T(S-P)}} = \lim_{P\to P_k}\frac{(P-P_k)\,X(P)}{1-e^{-T(S-P)}} \tag{119}$$

Particular values for parameter P_k, considering $k = 0,1,2,3,\cdots$, may be obtained by consulting (114).

So, we have for equation (119):

$$\lim_{P \to P_k} \frac{(P-P_k)\ X(P)}{1-e^{-T(S-P)}} = \left. \frac{(P_k-P_k)\ X(P)}{1-e^{-T(S-P)}} \right|_{P=S+j\frac{2\pi}{T}k} = \qquad (120)$$

$$= \frac{(P_k-P_k)\ X(P)}{1-1} = \frac{0}{0} = ind\ !$$

For solving foregoing equation, we have to proceed, first, by elimination of the specific indetermination condition by using L'Hospital rule that determines the application of derivative into limit operation.
Then, we can rewrite (120) as follows:

$$\mathrm{Re}\,sidues\ of\ \frac{X(P)}{1-e^{-T(S-P)}} = \lim_{P \to P_k} \frac{\frac{d}{dP}[(P-P_k)\ X(P)]}{\frac{d}{dP}[1-e^{-T(S-P)}]} = \qquad (121)$$

$$= \lim_{P \to P_k} \frac{X(P)+(P-P_k)\frac{d}{dP}\ X(P)}{-[-T(-1)]\ e^{-T(S-P)}} = \frac{X(P)}{-[-T(-1)]\ e^{-T(\frac{2\pi j}{T})}} = -\frac{1}{T}X(P)$$

With the solution in equation (121), we can obtain the solution for previous equation (117) and solving the integral formula, as follows:

$$X^*(s) = \frac{1}{2\pi j}\oint \frac{X(p)}{1-e^{-T(S-P)}}dp = \frac{-2\pi j \sum [\mathrm{Re}\,sidues\ of\ \frac{X(p)}{1-e^{-T(S-P)}}]}{2\pi j} = \qquad (122)$$

$$= \frac{-2\pi j}{2\pi j}\sum [\mathrm{Re}\,sidues\ of\ \frac{X(p)}{1-e^{-T(S-P)}}] = -\sum_{-\infty}^{\infty}[-\frac{1}{T}X(P)]$$

Finally, we can write the generic mathematical expression for Laplace transformation of sampled function $X^*(s)$ as a function of poles of impulsive modulation function.
Then we have:

$$X^*(s) = \frac{1}{T} \sum_{K=-\infty}^{\infty} [X(S + j\frac{2\pi}{T}k)] \quad (123)$$

It is interesting noticing that, if we want to work in the *z*-plane, only a changing of variable is necessary. Make it possible, we have:

$$X(z) = \frac{1}{T} \sum_{K=-\infty}^{\infty} [X(\frac{1}{T}\ln z + j\frac{2\pi}{T}k)] \quad (124)$$

In (124) we can notice that $s = \frac{1}{T}\ln z$ or that $e^{ST} = Z$.

Other remark, in the way of composition of original sampled function, during the reconstructed process, parameter k varies from $-\infty$ to $+\infty$ what corresponds to the modulation process.

10.1. SAMPLING THEOREM OPTIONAL CALCULUS

During the last generic demonstration, much information were obtained from a specific disposition of poles considering the polynomial ratio $\frac{X(P)}{1 - e^{-T(S-P)}}$.

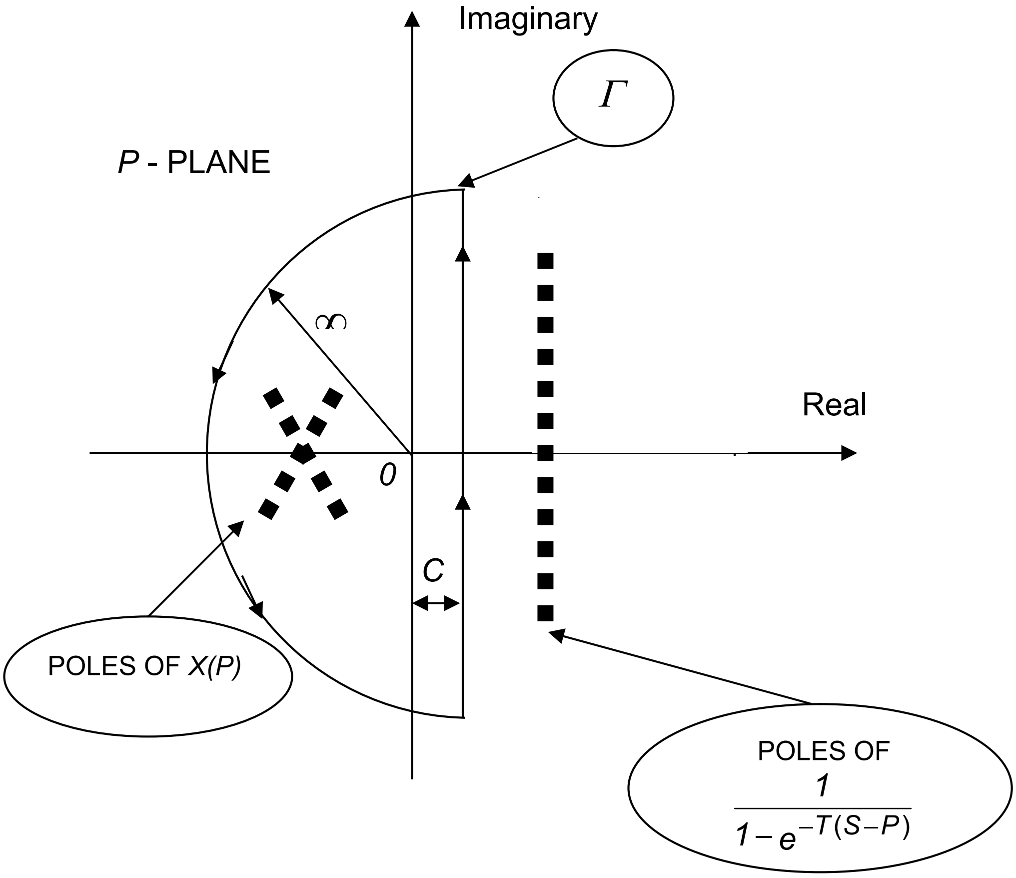


Figure 8 – Complex *P* - Plane 2

Now, demanding that the new contour be operating over the opposite region than the other considered in figure 7, we can obtain a new configuration as shown in figure 8, new figure for the *p*-plane 2.
Taking the same proposition that has been adopted for certain explanation, when the closed contour were oriented to right-half complex *p*-plane, we can rewrite equation (115), but now considering the closed contour oriented to left-half complex *p*-plane, as shown in figure 8.
So, we have:

$$X^*(s) = \frac{1}{2\pi j}\int_{c-j\infty}^{c+j\infty} \frac{X(P)}{1-e^{-T(S-P)}}dP = \\ = \frac{1}{2\pi j}\oint \frac{X(P)}{1-e^{-T(S-P)}}dP - \frac{1}{2\pi j}\int_{\Gamma} \frac{X(P)}{1-e^{-T(S-P)}}dP \qquad (125)$$

Generic equation for this case is exactly the same as (115), but for both analyses the considerations are different from each other.
Now, we can notice that, the second integral, integral acting over semicircle Γ, does not encircle any poles and, when the condition $P \to \infty$ is effectuated, denominator $1-e^{-T(S-P)}$ goes to unity as previous case, but function *X(P)* can admit and present values different from zero, so, $\lim_{P\to\infty} [P\, x(P)] \neq 0$. And then, integral acting over semicircle Γ is not always zero. This fact occurs because of the region at that the semicircle Γ is operating. Paying attention to this, we can firmly observe that, exactly inside this region, poles of function *X(P)* are found, what can bring a new consideration. If we admit, for consideration, a approximation that substitutes *X(P)* by $\lim_{\varepsilon\to 0} [e^{\varepsilon P} x(P)]$ with $(\varepsilon > 0)$.
Previous method of approximation can bring …
So, we obtain:

$$\frac{1}{2\pi j}\int_{\Gamma} \frac{X(P)}{1-e^{-T(S-P)}}dP = 0$$

and:

$$X^*(s) = \frac{1}{2\pi j}\oint \frac{X(P)}{1-e^{-T(S-P)}}dP \qquad (126)$$

By using previous concept and rewriting equation (118), we have:

$$\frac{1}{2\pi j}\oint \frac{X(P)}{1-e^{-T(S-P)}}dp = \sum \left[\text{Re}\,sidues\ of\ \frac{X(P)}{1-e^{-T(S-P)}}\ at\ poles\ of\ X(P)\right] \tag{127}$$

In (127) we consider the movement along the closed contour developed in counterclockwise, so that, the signal, in the integral operation, can be taking positive.

To obtain poles of function *X(P)* at the adequate form, we have to use Heaviside theorem or partial fraction expansion theorem that can be condensed as follows:

$$\begin{aligned} X^*(s) &= \frac{1}{2\pi j}\oint \frac{X(P)}{1-e^{-T(S-P)}}dP = \\ &= \sum \left[\text{Re}\,sidues\ of\ \frac{X(P)}{1-e^{-T(S-P)}}\ at\ poles\ of\ X(P)\right] = \\ &= \sum_{i=1}^{m} \frac{1}{(n_i-1)!}\frac{d^{n_i-1}}{dP^{n_i-1}}\left[(P-P_i)\frac{X(P)}{1-e^{-T(S-P)}}\right]_{P=P_i} \end{aligned} \tag{128}$$

If we want to work in the *z*-plane, only a changing of variable is necessary.
By substituting $Z = e^{ST}$ into last equation, we have:

$$X(z) = \sum_{i=1}^{m} \frac{1}{(n_i-1)!}\frac{d^{n_i-1}}{dP^{n_i-1}}\left[(P-P_i)\ X(P)\ \frac{Z}{Z-e^{TP}}\right]_{P=P_i} \tag{129}$$

In equation (129) we still can the last changing by substituting variable *P* by *S*, so:

$$\begin{aligned} X(z) &= \sum_{i=1}^{m} \frac{1}{(n_i-1)!}\frac{d^{n_i-1}}{dS^{n_i-1}}\left[(S-S_i)\ X(S)\ \frac{Z}{Z-e^{TS}}\right]_{S=S_i} = \\ &= \sum \left[\text{Re}\,sidues\ of\ \frac{X(S)Z}{Z-e^{TS}}\ at\ poles\ of\ X(S)\right] \end{aligned} \tag{130}$$

10.2. SAMPLING THEOREM CONCLUSIONS

After to present lots of concepts and taking the basic theory to understand the application of Shannon's theorem, or sampling theorem, we are going to continue the explanation and make conclusions about it.
For facility, we rewrite an excerpt of previous text used to give sequence in this technical article.

Sampling theorem states that sampling frequency $\omega_S = \frac{2\pi}{T}$, where T is sampling period, has to be greater than $2\omega_1$, taking into account figure 6. In other words, what is printed here is that all design applied for discrete-time control has to obey an important relationship between sampled frequency and maximum frequency associated to control process and this relationship can be describe as $\omega_S > 2\,\omega_1$.
We are going to choice the study that was made taking, first, a specific closed contour encircling poles of modulation function. In this case, we have to consider that input function *X(S)* have its poles lain on the right-half complex *s*-plane. As input function is modulated by a signal at the frequency $\omega_S = \frac{2\pi}{T}$, and its harmonics, it is easy to notice that, in previous equation (123), sampled frequency may be expressed by the expression $\frac{2\,\pi}{T}k$. So, we can write, for sampled signal, the following equation:

$$X^*(s) = \frac{1}{T}\sum_{K=-\infty}^{\infty}[X(S + j\,\omega_S k)] = \frac{1}{T}\sum_{K=-\infty}^{\infty}X[j(\omega + \omega_S k)] \qquad (131)$$

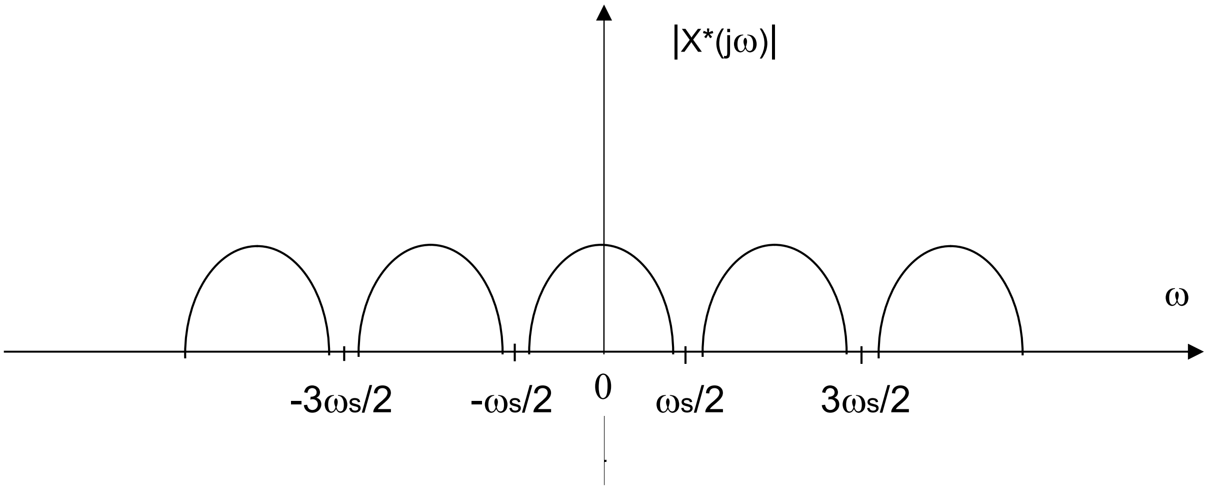
If we take the frequency spectrum for *X*(s)*, we have:

$$\left|X^*(s)\right| = \frac{1}{T}\left|\sum_{K=-\infty}^{\infty}X[j(\omega + \omega_S k)]\right| \qquad (132)$$

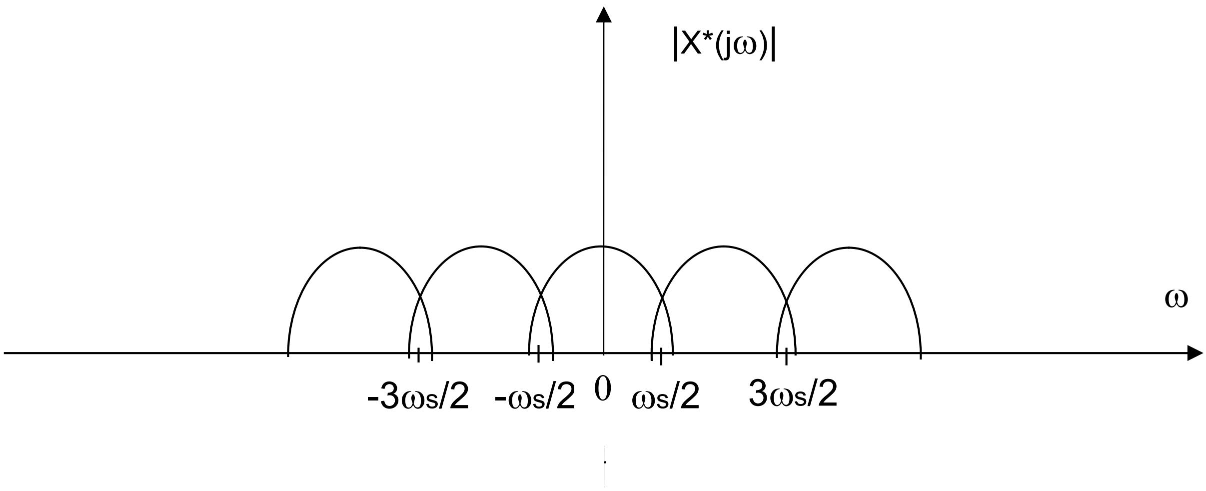
As there is a modulation procedure, we are going to develop last equation in expansion of steps considering each module.
So, we have:

$$\left|X^*(s)\right| = \cdots + \frac{1}{T}\left|X[j(\omega - \omega_S)]\right| + \frac{1}{T}\left|X(j\omega)\right| + \frac{1}{T}\left|X[j(\omega + \omega_S)]\right| + \cdots \qquad (133)$$

Now, representation of expanded summation, seen previously, must consider two conditions for the value taking to sampled frequency in relation to maximum frequency that can be observed from the motion of control process. We have to remember that maximum frequency of process is always associated to minimum time constant of this process. So, the reconstruction of the sampled signal involved here has to consider this appointment.



(a) Spectrum for ωs > 2ω1



(b) Spectrum for ωs < 2ω1

Figure 9 – Spectrum of |*X*(*jω*)| as a Function of ω

When we observe figure 6, that represents the frequency response of continuous signal input *x*(t)*, it is clearly that limits $-\omega/2$ and $+\omega/2$ are, in fact, the field of existence taking the frequency spectrum as a reference. If it is necessary to reconstruct the whole

continuous signal, we have to worry about the range that must be respected for that signal being represented. This frequency range is given by limits $-\omega/2$ and $+\omega/2$, mentioned previously because, within this range, the signal performance will be respected without being bypassed or overlapped.
Now, it is clearly too when there is a processing in course that has to promote all the reconstructed step by step, considering the modulation process, each piecewise of time will determine whether there has been a completely reconstruction or not.
Figure 9 shows that each plot of $|X^*(j\omega)|$ versus ω consists of $|X(j\omega)|$ repeated every sampled frequency $\omega_S = \frac{2\pi}{T}$ Radian per second.
Paying attention to preceding figure 9, that shows the spectrum for two conditions, we can see important conclusions about the particular value for parameter *T*. We may have in mind that parameter *T* is directly linked to sampling process or more precisely linked to time constants involved with the process.
For the condition where $\omega_S > 2\omega_1$ or $T < \frac{\pi}{\omega_1}$, taking into account figure 9 (a), we can notice that there is no overlap effect over complementary components of primary component. Thus, the original shape of primary component is preserved by the sampling process.
Now, for the opposite condition where $\omega_S < 2\omega_1$ or $T > \frac{\pi}{\omega_1}$, taking into account figure 9 (b), we can notice that there is overlap effect acting over each components. Thus here the original shape of primary component is not preserved by the sampling process anymore.
The reconstruction of the original signal, if the condition $T < \frac{\pi}{\omega_1}$ is respected, it is possible by using low-pass filter exactly after the signal having been sampled.
So, Shannon's theorem can be proved by lots of insights shown the important relationship between sampling frequency and the one that characterize the motion of the process associated to input continuous signal.

10.3. SAMPLING THEOREM OBSERVATIONS

It is very important to understand the effect introduced into sampling process by sources of error that we can specified based on the distortion what does not allow a perfect reconstruction when the synthesizing process is activated. There are two sources of error and we can take the following explanation about it.
Taking following equation from equation (123), that was developed previously, we can obtain:

$$X^*(j\omega) = \frac{1}{T}\sum_{K=-\infty}^{\infty}[X(j\omega + j\frac{2\pi}{T}k)] = \frac{1}{T}\sum_{K=-\infty}^{\infty}[X\, j(\omega + k\,\omega_S)] \qquad (134)$$

Equation (134) has already been understood as a proper relationship in which appears the influence of sampling frequency $\omega_S = \frac{2\ \pi}{T}$, used here into this procedure.

It is clearly that, first; the sampling process introduces lots of distortion into the frequency spectrum of continuous signal that has been sampled. This kind of error is only associated to sampling process prior to the reconstruction and synthesizing step.
Another error can be understood as a reconstruction error that can be introduced by phase shift from lag effects that are associated to lag time constants involved with the process, mainly during closing loop action applied to feedback automatic control.
For us, the focus is on the first problem associated to the distortions caused by the sampling process.
Taking into account equation (134), we are going to develop two situations as follows.
Imagine the input function *X(S)* is a sine wave at 1 Hz and sampling frequency is performed at 10 Hz. As sampling frequency is greater than process frequency we do not have more problems with the sampled signal. Frequency components verified into spectrum sequence are 1 Hz, 11Hz, 21 Hz, 31 Hz, 41 Hz, 51 Hz, … To obtain this we can take equation (134) with $\omega = 1$ and $\omega_S = k\,[1]$ considering *K*= *0, 1, 2, 3, …*

Now, imagine the input function *X(S)* is a sine wave at 11 Hz and sampling frequency is performed at the same 10 Hz. As sampling frequency is fewer than process frequency we have problems of distortions on the spectrum frequency graphics. At that now situation frequency components verified into spectrum sequence are 21 Hz, 31 Hz, 41 Hz, 51 Hz, …

X

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